A Simple Proof of Pascal's Hexagon Theorem

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Jan van Yzeren

Pascal's Theorem. *If the vertices of a hexagon lie on a circle and the three pairs of opposite sides intersect, then the three points of intersection are collinear.*

This theorem was published in 1640 by sixteen-year-old Blaise Pascal. His original proof has been lost, and at times one wonders whether one or another of the known proofs is, in fact, Pascal's original one. This also applies to the simple proof given here.

Begin with the hexagon A_i, $i = 0, \dots, 5$ of Figure 1, and consider the circle through the points A_1, A_4 and P_1, where the first two points are (opposite) vertices, and the last is one of the "Pascal points" connected to them. This circle meets A_0A_1 and A_3A_4 at B_0 and B_1 respectively, and one uses arcs of the circles shown to find equal angles inscribed in them (or supplementary angles inscribed in opposite arcs). As a consequence, the triangles $P_1B_0B_1$ and $P_2A_0A_3$ have respectively parallel sides, that is, they are perspective from the point P_0. Therefore, P_0, P_1 and P_2 are collinear.

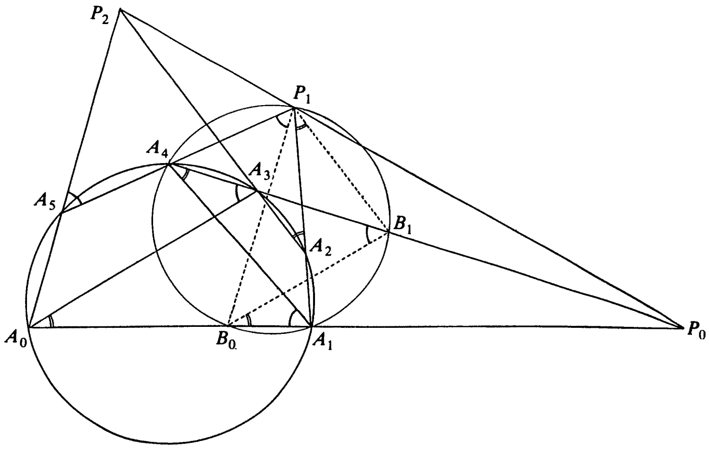

Figure 1

The proof also covers the case of $A_0A_1 \| A_3A_4$ (i.e., P_0 at infinity). Then, the triangles are translative, that is, P_1P_2 is parallel with A_0A_1 and A_3A_4. The only special case not covered by the proof concerns hexagons inscribed in a circle with parallels as opposite sides. This case, however, follows easily from appropriate arcs.

Whether Pascal gave this proof is open to debate, but it seems that this proof has not turned up for 350 years. On this point Professor Coxeter kindly has commented as follows: "It is indeed remarkable that this elegant proof was not found in 350 years, and also somewhat remarkable that Guggenheimer came close to it in 1967 and then felt obliged to introduce a peculiar lemma." [3]

Anyway, the historic delay justifies some special attention for the heuristics of this simple proof.

The basic figure consists of two pencils of four lines joining points on a circle, viz. (Figure 2) A_0 and A_4 with, respectively, A_1, A_2, A_3 and A_5.

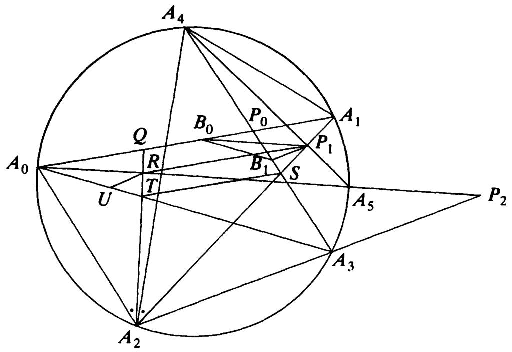

Figure 2

Evidently, the two pencils are congruent (equal angles between corresponding lines). Therefore, if $\triangle A_0A_2Q$ is made similar to $\triangle A_4A_2A_1$, the segments A_2Q and A_2A_1 are divided proportionally and $A_1A_0//P_1R//ST$. Now, the crucial idea is to build up this basic figure in a converse manner, starting with two given similar triangles: $\triangle A_0A_2Q \sim \triangle A_4A_2A_1$ and forgetting the circle.

Then, choose P_1 and R on, respectively, A_1A_2 and QA_2, such that $P_1R//A_1A_0$. Similarly S and T. Hereupon the following points are defined: $A_5 = A_4P_1 \cap A_0R$, $A_3 = A_4S \cap A_0T$, $P_0 = A_4S \cap A_0A_1$, $P_2 = A_0R \cap A_2A_3$.

To prove that P_0, P_1 and P_2 are collinear:

Consider $\triangle RA_0U$, $RU//P_2A_3$, and its translative image $\triangle P_1B_0B_1$. Then, B_0 lies on P_0A_0 as $P_0A_0//P_1R$, and B_1 lies on P_0A_3, because $P_1B_1 = RU = A_2A_3 \cdot RT/A_2T = A_2A_3 \cdot P_1S/A_2S$. Therefore, the triangles $P_1B_0B_1$ and $P_2A_0A_3$ are perspective from the point P_0 and, indeed, P_0, P_1 and P_2 are collinear.

Afterwards the crucial points B_0 and B_1 can be found directly. In fact, they lie on the circumcircle of $\triangle P_1A_1A_4$, because $\angle P_1B_0A_1 = \angle A_5A_0A_1 = \angle A_5A_4A_1 = \angle P_1A_4A_1$ and $\angle A_4B_1B_0 = \angle A_4A_3A_0 = \angle A_4A_1A_0 = \angle A_4A_1B_0$. Actually, drawing the circumcircle of $\triangle P_1A_1A_4$ is the very point of the new proof.

Background of the heuristics is the fact that the metric of the Euclidean plane can be defined by giving a pair of similar triangles. After that, all other metric properties must follow by means of parallels and proportionalities (affine tools).

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